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DYNAMICS AND STABILITY IN AN OLG MODEL WITH NON-SEPARABLE PREFERENCES

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1. Introduction

In the literature on steady state stability in OLG models, Diamond (1965) model represents the benchmark model with separable preferences. Under the assumption of separable preferences, de la Croix (1996) proves that the optimal solution in the Diamond (1965) model is always characterised by monotonic convergence to the steady state. Michel and Venditti (1997) and de la Croix and Michel (1999) both depart from Diamond (1965) model assuming non-separable preferences: Michel and Venditti (1997) provide sufficient conditions for stability of the equilibrium in an OLG model with separable preferences across generations only and prove that the optimal solution may be oscillating and optimal cycles may exist; de la Croix and Michel (1999) provide sufficient conditions for existence and uniqueness of the equilibrium in an OLG model with separable preferences within generations only and prove that the optimal solution may display damped oscillations even when the social planner does not discount the utility of future generations.[1]

The main contribution of this paper is to provide sufficient conditions for existence and uniqueness of a steady state equilibrium in a production economy of overlapping generations with non-separable preferences, i.e. habits are transmitted from one generation to the next one (intergenerational spillover) *and* from one period to the next one (intragenerational spillover), and to analyse the implications of non-separable preferences (*both* across and within generations) for the local stability of the steady state equilibrium. The present OLG model is by no means trivial as it provides completely new results in terms of dynamics and stability under the assumption of non-separable preferences, both across and within generations, and it therefore fills in a gap in the existing literature.

The rest of the paper is organised as follows. In section 2 we describe the model, in section 3 we analyse the competitive setting, while in section 4 we analyse the optimal solution and derive conditions under which the optimal solution is stable.

[1] Both Michel and Venditti (1997) and de la Croix and Michel (1999) assume a production economy of overlapping generations of agents. Two references for dynamics and stability in a pure exchange economy of overlapping generations agents with non- separable preferences are instead Lahiri and Puhakka (1998) and Orrego (2014).

We prove that, under the assumption of intergenerational and intragenerational spillovers, convergence of the optimal solution to the non-trivial steady state is not always assured and that the optimal solution may display either locally explosive dynamics or damped oscillations. We provide concluding remarks in section 6.

2. The model

The model is a simple extension of the Diamond (1965) economy without outside money. At each date, the economy is populated by three generations (young, adult and old), each living for three periods. The growth rate of population is zero. The young generation has no decision to take and only inherits habits h_t from the previous adult generation according to the following equation

$$h_t = c_{t-1}^a \tag{1}$$

where c_{t-1}^a is the consumption of the adult generation at time $t-1$. The adult generation draws utility from consumption of the quantity c_t^a, given its own stock of habits h_t. When old, each agent draws utility from consumption of the quantity c_{t+1}^o, given her own past consumption c_t^a. The intertemporal utility function of each adult agent is

$$U(c_t^a, c_{t+1}^o; h_t) = u(c_t^a - \theta h_t) + v(c_{t+1}^o - \delta c_t^a) \tag{2}$$

where $\theta \in (0,1)$ measures the intensity of the intergenerational spillover effect due to the inherited habits (passive effect) and $\delta \in (0,1)$ measures the intensity of the intragenerational spillover effect due to the persistence of own preferences over time (active effect). In other words, we assume that adult consumption at time $t-1$ determines a frame of reference against which adult individual consumption at time t is judged and that the depreciation rate of these inherited habits is so high that it no longer affects the evaluation of consumption when old. We also assume that adult consumption at time t determines a frame of reference against which old individual consumption at time $t+1$ is judged and that persistence of preferences is so high that neither young consumption at time t nor that at time $t-1$ affect in any possible way the evaluation of consumption when old.

Moreover we assume that the utility function is strictly increasing with respect to consumption and decreasing with respect to the stock h: $u_{c^a} > 0$, $v_{c^o} > 0$, $u_h < 0$, $u_{c^a c^a} < 0$, $v_{c^o c^o} < 0$, $u_{hh} < 0$ and $u_{c^a h} > 0$. The assumption $u_{c^a h} > 0$ amounts to postulating that an increase in the stock h rises the desire for consumption. We also assume that starvation is ruled out in both periods

$$\lim_{c_t^a \to 0} u_{c^a} + v_{c^a} = \lim_{c_{t+1}^o \to 0} v_{c^o} = \infty \tag{3}$$

and that the utility function is strictly concave under the following condition

$$\delta < \frac{u_{c^a}}{v_{c^o}} \tag{4}$$

Note that if preferences are separable as in Diamond (1965), $\delta = 0$ and strictly concavity is always ensured by the standard set of assumptions on marginal utility, i.e. $u_{c^a} > 0$ and $v_{c^o} > 0$. If preferences are non- separable, $\delta > 0$ and concavity is ensured only if condition (4) holds. Otherwise, the utility function is flat ($\delta = \frac{u_{c^a}}{v_{c^o}}$) or convex ($\delta > \frac{u_{c^a}}{v_{c^o}}$).

At each date a single good is produced. This good can be either consumed or accumulated as capital for future production. Production occurs through a constant return to scale technology. Per capita output y_t is a function of capital intensity k_t

$$y_t = f(k_t) \tag{5}$$

in which $f(\ )$ is a neoclassical production function with $f_k > 0$ and $f_{kk} < 0$. Assuming total depreciation of capital after one period, the resource constraint of the economy is

$$y_t = c_t^a + c_t^o + k_{t+1} \tag{6}$$

At date 0 the economy is endowed with a given quantity of capital per capita k_0 and a level of inherited habits h_0.

3. The competitive economy

The competitive behaviour of firms leads to the equalisation of the marginal productivity of each factor to its marginal cost:

$$R_t = f_k(k_t) \tag{7}$$
$$w_t = f(k_t) - k_t f_k(k_t) \tag{8}$$

where R_t is the interest factor paid on loans and w_t is the real wage paid to workers.

The adult generation works during the period t and sells one unit of labor inelastically at any real wage w_t, consumes the quantity c_t^a and saves s_t for the next period by holding capital

$$c_t^a = w_t - s_t \tag{9}$$

while the old generation spends all her savings s_t plus interest matured and consumes c_{t+1}^o

$$c_{t+1}^o = R_{t+1} s_t \tag{10}$$

The maximisation program of each individual is thus to choose c_t^a, c_{t+1}^o in order to

$$\max_{c_t^a, c_{t+1}^o} u(c_t^a - \theta h_t) + v(c_{t+1}^o - \delta c_t^a)$$

subject to $c_t^a = w_t - s_t$, $c_{t+1}^o = R_{t+1} s_t$

where w_t, $R_{t+1}$and h_t are given to the agent. Assuming an interior solution, under rational expectations, the above decision problem leads to following first order condition

$$u_{c^a} - \delta v_{c^a} = R_{t+1} v_{c^o} \quad (11)$$

With respect to a standard Diamond (1965) model in which $\delta = 0$, marginal utility of the young is lower, as $v_{c^o} > 0$: in order to achieve the same level of satisfaction when old, adults need to correct their satisfaction by the (negative) habit effect.

Equation (11) allows to define the following saving function

$$s_t = s(w_t, R_{t+1}, h_t) \quad (12)$$

The partial derivative of the saving function (12) are

$$s_w = \frac{u_{c^a c^a}}{u_{c^a c^a} + R_{t+1}^2 v_{c^o c^o}} > 0, s_r = \frac{-(v_{c^o} + v_{c^o c^o} c_{t+1}^o)}{u_{c^a c^a} + R_{t+1}^2 v_{c^o c^o}}, s_h = \frac{-\theta u_{c^a h}}{u_{c^a c^a} + R_{t+1}^2 v_{c^o c^o}} < 0$$

Since the utility function is concave and there is no wage income in the last period of life, savings increase with wage income. The effect of the interest rate is instead ambiguous and depends on the value of the intertemporal elasticity of substitution, $\frac{v_{c^o c^o} c_{t+1}^o}{v_{c^o}}$. Finally, the effect of rising inherited habits is negative: when the passive effect is low, the agent has a sober lifestyle and savings are high; when the passive effect is high, the agent spends much on consumption to maintain a life standard similar to the one of their peers and their propensity to save is low.

The equilibrium condition in the capital market implies

$$k_{t+1} = s_t \quad (13)$$

Combining equations (1), (7), (8), (12) and (13), the competitive equilibrium is defined as a sequence $\{k_t, h_t; t > 0\}$ which satisfies

$$k_{t+1} = s(f(k_t) - k_t f_k(k_t), f_k(k_{t+1}), h_t) \quad (14)$$

$$h_{t+1} = f(k_t) - k_t f_k(k_t) - s(f(k_t) - k_t f_k(k_t), f_k(k_{t+1}), h_t) \quad (15)$$

Equation (14) is the clearing condition of the asset market, given that the labour market is in equilibrium (i.e. that (8) holds). It reflects the fact that savings are to be equal to the capital stock of the next period. Equation (15) is the equation (1), given that the asset and the labour markets are in equilibrium.

As in de la Croix and Michel (1999), the competitive equilibrium is characterised by spillovers from one generation to the next and from adulthood to old age. The main components of the intergenerational spillovers are: savings by old people and past consumption levels of the previous generation. While the process transforming the savings by the old into income for the adult displays decreasing returns to scale due to the characteristics of the production function, the process transforming past consumption of the adults into consumption of the next generation displays constant returns to scale due to the characteristics of the utility function. The intragenerational spillovers is only given by the individual past consumption that feeds individual's habits from adulthood to old age. This process displays constant returns to scale, again due to the characteristics of the utility function. Thus, even though the intergenerational bequest in terms of higher wages will not be sufficient to cover the intergenerational bequest in terms of higher inherited habits, the intragenerational spillover leaves a bequest in terms of higher persistence. The combination of the positive bequests in terms of higher wages and higher persistence is sufficient to offset the negative bequest in terms of the higher externality. This leads to an increase in saving to maintain future standards of consumption that induces an expansion. When the enrichment is strong enough, the externality has already reverted to higher levels, allowing a fall in savings and the start of a recession. As the effect due to persistence is stronger than the effect due to the externality, the model is characterised by converging cycles. Thus, the competitive equilibrium still displays fluctuations, but the bifurcation corresponds to different critical values of θ and δ. Depending on the parameters θ and δ, the economy may converge to or diverge from the steady state.

As such results are perfectly in line with the existing literature on growth when preferences are transmitted from one generation to another (de la Croix and Michel, 1999), the rest of the paper focuses on the optimal solution and derives conditions under which the sub-optimality and instability caused by the externalities (intergenerational and intragenerational spillovers due to non-separable preferences) can be overtaken.

4. The optimal solution

As inherited habits introduce an externality in the model, the decentralised equilibrium is obviously sub- optimal compared to the equilibrium that would maximise the planner's utility. Thus, hereafter, we focus our attention to the optimal solution and consider a social planner who chooses the allocation of output in order to maximise the present discount value of current and future generations.

Assuming that the social planner's discount factor is γ, the social planner maximisation program is thus to choose $\{c_t^a, c_t^o\}$ and $\{k_t, h_t\}$ in order to

$$\max_{c_t^a, c_t^o, k_t, h_t} \sum_{t=0}^{\infty} \gamma^t \left[u(c_t^a - \theta h_t) + \frac{1}{\gamma} v(c_t^o - \delta c_t^a) \right] \tag{16}$$

$$\text{subject to} \begin{array}{l} y_t = c_t^a + c_t^o + k_{t+1} \\ h_t = c_{t-1}^a \end{array}$$

and given k_0 and h_0. First order conditions are

$$u_{c^a}(c_t^a - \theta h_t) + \gamma u_h(c_{t+1}^a - \theta h_{t+1}) = \frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) - v_h(c_{t+1}^o - \delta h_{t+1}) \tag{17}$$

$$\frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) = v_{c^o}(c_{t+1}^o - \delta h_{t+1}) f_k(k_{t+1}) \tag{18}$$

Equation (17) is a condition for optimal intergenerational allocation of consumption between adult and old alive at the same time. Marginal utility of the adult, corrected by the social planner to internalise the taste externality, is equalised to marginal utility of the old is equal to the marginal utility of the old. Note that, due to the presence of the taste externality and contrary to the standard Diamond (1965) model, this social planner's first order condition does not respect the individual first order condition (11). Moreover, with respect to the standard Diamond (1965) model in which $\delta = \theta = 0$, marginal utility of the adult, $u_{c^a}(c_t^a - \theta h_t) + \gamma u_h(c_{t+1}^a - \theta h_{t+1})$, is lower, as $u_h < 0$, while marginal utility of the old, $\frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) - v_h(c_{t+1}^o - \delta h_{t+1})$, is higher, as $v_h < 0$. Equation (18) sets the optimal intertemporal allocation.

The optimal equilibrium is defined as a sequence $\{c_t^a, c_t^o, k_t, h_t; t > 0\}$ which satisfies equations (17), (18), (6) and (1) simultaneously:

$$u_{c^a}(c_t^a - \theta h_t) + \gamma u_h(c_{t+1}^a - \theta h_{t+1}) = \frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) - v_h(c_{t+1}^o - \delta h_{t+1}) \tag{19}$$

$$\frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) = v_{c^o}(c_{t+1}^o - \delta h_{t+1}) f_k(k_{t+1}) \tag{20}$$

$$h_t = c_{t-1}^a \tag{21}$$

$$k_{t+1} = f(k_t) - c_t^a - c_t^o \tag{22}$$

It appears from the system above that the steady state (c^a, c^o, k, h) of this optimal economy is defined by

$$u_{c^a}(c^a - \theta h) + \gamma u_h(c^a - \theta h) = \frac{1}{\gamma} v_{c^o}(c^o - \delta h) - v_h(c^o - \delta h) \tag{23}$$

$$\frac{1}{\gamma} = f_k(k_{t+1}) \tag{24}$$

$$h = c^a \tag{25}$$

$$k = f(k) - c^a - c^o \tag{26}$$

Equation (23) shows that in an economy with passive habits, the marginal utility of the adult is lower than the corresponding marginal utility in the standard Diamond (1965) model: the inheritance represents a benchmark from which individuals want to depart. Even the marginal utility of the old is higher than the corresponding marginal utility in the standard Diamond (1965) model: the same interpretation carries on. Once the externality associated with parents' habits is internalised, persistence affects marginal utility in the same way as the externality: they both induce consumers to save. Equation (24) is the modified golden rule: the introduction of intergenerational and intragenerational spillovers does not modify the optimal steady- state stock of capital which remains fixed at the modified golden rule level. Equations (25) and (26) have been already discussed in the paper.

Proposition 1. *A positive steady state equilibrium exists and is unique if and only if* $\det(\mathbf{I} - \mathbf{J}^{SO}) \neq 0$, *where* $\mathbf{J}^{SO}$ *is the Jacobian matrix associated to the optimal equilibrium (19)-(22) and evaluated at steady state* (c^a, c^o, k, h).

Proof. See Appendix A.1.

Proposition 2. *Assume that* k *and* h *are state variables and that* c^a *and* c^o *are jump variables. Locally explosive dynamics is possible, depending on the sign of the trace* $\mathbf{T}_{\mathrm{J}}^{SO}$ *and of the element* Z *of the Jacobian matrix* $\mathbf{J}^{SO}$. *If* $\Delta \geq 0$, *the eigenvalues are real and local dynamics is either explosive or monotonic. If* $\Delta < 0$, *the eigenvalues are complex and conjugate and local dynamics displays either explosive or damped oscillation.*

Proof. See Appendix A.2.

The above proposition identifies all possible dynamics of the optimal steady state equilibrium. Under the assumption that the trace $\mathbf{T}_{\mathrm{J}}^{SO}$ and the element Z are both positive, locally explosive dynamics is identified by an unstable node if the eigenvalues are real and by an unstable focus if the eigenvalues are complex and conjugate. Under the assumption that $\mathbf{T}_{\mathrm{J}}^{SO}$ and Z are both negative, the optimal solution is a stable saddle point, only if the constraints on the elements of the Jacobian matrix $\mathbf{J}^{SO}$ respect the condition on negativity of the trace. Under the assumption that $\mathbf{T}_{\mathrm{J}}^{SO}$ and Z have opposite sign, the optimal solution may be either stable or unstable: if stable, dynamics displays damping oscillation to the steady state; if unstable, locally explosive dynamics occurs when the constraints on the elements of the matrix $\mathbf{J}^{SO}$ do not respect the condition on negativity of the trace.

The stability of the optimal steady state equilibrium depends on the assumption that habits are trans- mitted both across and within generations, assumption that affects the sign of the trace $\mathbf{T}_{\mathrm{J}}^{SO}$ and of element Z. Monotonic convergence to the optimal steady state equilibrium is ensured only under the assumption that the stock of inherited habits does not persist into their old age, i.e. only if $\delta = 0$ as in the standard Diamond (1965) model. Contrary to the competitive equilibrium, the optimal solution is only characterised by a positive intragenerational spillover: savings by the old, that directly finance the capital stock required for production in the next period and indirectly sustain wages of the adult. The intergenerational spillover due to habits is *a priori* internalised by the social planner in the maximisation problem (16). As in the competitive equilibrium, the process transforming the savings by the old into income for the adult displays decreasing returns to scale, due to the characteristics of the production function. However, the intergenerational bequest in terms of higher wages does not interact with any other spillover. The intergenerational bequest in terms of higher wages will lead to a constant increase in saving that induces a permanent expansion. The model might thus be characterised by diverging explosive dynamics. A numerical example is provided in Marini (2015).

5. Conclusions

This paper derives sufficient conditions for existence of a steady state equilibrium in an OLG model with non-separable preferences and analyses the implications of non-separable preferences for the local stability of the steady state equilibrium.

It studies conditions for the existence and stability of optimal equilibrium and it proves that the optimal solution may display damped oscillations or locally explosive dynamics. This result crucially depends on the assumption that habits are transmitted from one generation to the next one and from adulthood to old age.

This paper shows that combining different forms of non-separable preferences is not innocuous: when we introduce persistence of individual tastes in the contest of an OLG model in which habits are inherited, dynamics of the model and stability of the equilibrium are considerably affected. The results presented in this paper are therefore fundamental to understanding the mechanisms underneath models with habit formation and habit persistence, as habits seem to play a significant role in many aspects of economic theory.

Appendix

A.1 Proof of Proposition 1

First, linearize the non linear dynamic system (19)-(22) around the steady state (23)-(26) and get

$$\begin{bmatrix} \mathrm{d}h_{t+1} \\ \mathrm{d}c_{t+1}^{a} \\ \mathrm{d}k_{t+1} \\ \mathrm{d}c_{t+1}^{o} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{[B(1-\gamma E)-1]}{\gamma} & A-B(D-E) & \dfrac{DB}{\gamma} & C-B(1+D) \\ 0 & -1 & \dfrac{1}{\gamma} & -1 \\ E & D-E & -\dfrac{D}{\gamma} & 1+D \end{bmatrix} \begin{bmatrix} \mathrm{d}h_{t} \\ \mathrm{d}c_{t}^{a} \\ \mathrm{d}k_{t} \\ \mathrm{d}c_{t}^{o} \end{bmatrix}$$

in which $A \equiv -\frac{u_{c^a c^a}+\gamma u_{hh}+v_{hh}}{\gamma u_{c^a h}} > 0$, $B \equiv \frac{v_{c^o h}}{\gamma u_{c^a h}} > 0$, $C \equiv \frac{v_{c^o c^o}}{\gamma^2 u_{c^a h}} < 0$, $D \equiv \frac{\gamma v_{c^o} f_{kk}}{v_{c^o c^o}} > 0$, $E \equiv \frac{v_{c^o h}}{v_{c^o c^o}} < 0$ under the assumption that in steady state $f_k(k_{t+1}) = f_k(k_t) = \gamma^{-1}$. As the Jacobian matrix evaluated at steady state (c^a, c^o, k, h) is

$$\mathbf{J}^{\mathrm{SO}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{[B(1-\gamma E)-1]}{\gamma} & A-B(D-E) & \dfrac{DB}{\gamma} & C-B(1+D) \\ 0 & -1 & \dfrac{1}{\gamma} & -1 \\ E & D-E & -\dfrac{D}{\gamma} & 1+D \end{bmatrix}$$

it is immediate to show that the matrix $\mathbf{I}-\mathbf{J}^{\mathrm{so}}$ is equal to

$$\mathbf{J}^{\mathrm{SO}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -\dfrac{[B(1-\gamma E)-1]}{\gamma} & 1-A+B(D-E) & -\dfrac{DB}{\gamma} & -C+B(1+D) \\ 0 & 1 & 1-\dfrac{1}{\gamma} & 1 \\ -E & -D+E & \dfrac{D}{\gamma} & -D \end{bmatrix}$$

and that its determinant is

$$\det(\mathbf{I}\text{-}\mathbf{J}^{\mathrm{SO}}) = (A+B-C)D - D - \frac{(1-B)D}{\gamma} = D\left[(A+B-C)+\frac{B}{\gamma}-\frac{1+\gamma}{\gamma}\right]$$
$$= \frac{\gamma v_{c^o} f_{kk}}{v_{c^o c^o}}\left[\frac{v_{c^o h}-(u_{c^a c^a}+\gamma u_{hh}+v_{hh})}{\gamma u_{c^a h}} + \frac{v_{c^o h}-v_{c^o c^o}}{\gamma^2 u_{c^a h}} - \frac{1+\gamma}{\gamma}\right] \neq 0$$

under the assumptions that the utility function is concave, that the production function is neoclassical, that equations (23)-(26) hold and that conditions (25) and (26) are met.

A.2 Proof of Proposition 2

The characteristic polynomial P in the eigenvalues σ associated to the Jacobian matrix $\mathbf{J}^{\mathrm{SO}}$ evaluated at steady state (c^a, c^o, k, h) is

$$P(\sigma) = \sigma^4 - \mathrm{T}_{\mathrm{J}}^{\mathrm{SO}}\sigma^3 + Z\sigma^2 - \gamma^{-1}\mathrm{T}_{\mathrm{J}}^{\mathrm{SO}}\sigma + \det(\boldsymbol{J}^{SO}) = 0$$

in which

$\det(\boldsymbol{J}^{SO}) = \frac{1 - B + \gamma CE}{\gamma^2} = \frac{1}{\gamma^2}$
$\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}} = 1 + \gamma^{-1} + A - B(D - E) + D \gtreqless 0$
if $1 + \gamma^{-1} + A + D \gtreqless B(D - E)$
$Z = 2\gamma^{-1} + (1 + \gamma^{-1} + A - B(D - E) + D) \gtreqless 0$
if $1 + 3\gamma^{-1} + A + D \gtreqless B(D - E)$

In order to study the polynomial P, factorize the polynomial P into
$P(\sigma) = (\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3)(\sigma - \sigma_4) = 0$
which is equivalent to

$$\left(\sigma^2 - \phi_1\sigma + \frac{1}{\gamma}\right)\left(\sigma^2 - \phi_2\sigma + \frac{1}{\gamma}\right)$$

in which $\phi_1 \equiv \sigma_1 + \sigma_2$ and $\phi_2 \equiv \sigma_3 + \sigma_4$. Then analyse all possible scenarios, due to the sign's ambiguity of the trace $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}}$and of element Z.

First, assume that $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}}$and Z are both positive and analyse the two possible cases:

1. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geq 0$, $i = 1,2$. The four eigenvalues are real and they can be:
   a. four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}} > 0$.
   b. two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is excluded as it violates $Z > 0$.
   c. four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is accepted as it respects both conditions on $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}}$ and Z.
2. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0$, $i = 1,2$. Look at the real parts only. Since the real part $a = -\frac{1}{2}\phi_i \neq 0$, $i = 1,2$, the eigenvalues are complex and conjugate and they can be:
   a. four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}} > 0$.
   b. two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is excluded as it violates $Z > 0$.
   c. four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is accepted as it respects both conditions on $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}}$ and Z.

Under the assumption that $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}}$ and Z are both positive, the only admissible case is (c). It identifies an unstable node if the eigenvalues are real and an unstable focus if the eigenvalues are complex and conjugate. Locally explosive dynamics is highly likely.

Then, assume that on $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}}$ and Z are both negative and analyze the two possible cases:

1. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geq 0$, $i = 1,2$. The four eigenvalues are real and they can be:
   a. four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}}$ and Z.
   b. two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 < 0$ as $\mathbf{T}_{\mathrm{J}}^{\mathrm{SO}} < 0$.
   c. four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates condition on Z.

2. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0$, $i = 1{,}2$. Look at the real parts only. Since the real part $a = -\frac{1}{2}\phi_i \neq 0$, $i = 1{,}2$, the eigenvalues are complex and conjugate and they can be:
   a. four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_J^{SO}$ and Z.
   b. two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 < 0$ as $\mathbf{T}_J^{SO} < 0$.
   c. four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates Z.

Under the assumption that $\mathbf{T}_J^{SO}$ and Z are both negative, the only admissible case is b), but only if $\phi_1 + \phi_2 < 0$. It identifies a stable saddle point that ensures monotonic local convergence.

Finally, assume that $\mathbf{T}_J^{SO}$ and Z have opposite sign and distinguish two possible cases:

1. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geq 0$, $i = 1{,}2$. The four eigenvalues are real and they can be:
   a. four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_J^{SO}$ and Z.
   b. two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 > 0$ as it ensures $\mathbf{T}_J^{SO} > 0$.
   c. four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is admissible only if $\mathbf{T}_J^{SO} < 0$ and $Z > 0$.
2. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0$, $i = 1{,}2$. Look at the real parts only. Since the real part $a = -\frac{1}{2}\phi_i \neq 0$, $i = 1{,}2$, the eigenvalues are complex and conjugate and they can be:
   a. four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_J^{SO}$ and Z.
   b. two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 > 0$ as it ensures $\mathbf{T}_J^{SO} > 0$.
   c. four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is admissible only if $\mathbf{T}_J^{SO} < 0$ and $Z > 0$.

Under the assumption that $\mathbf{T}_J^{SO}$ and Z have opposite sign, case (b) identifies an unstable solution as $\mathbf{T}_J^{SO} > 0$. Locally-explosive dynamics is highly likely. Case (c) identifies a stable node for real eigenvalues and a stable focus for complex and conjugate eigenvalues, and therefore it ensures damped convergence to the steady state.

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SUMMARY

Dynamics and stability in an OLG model with non-separable preferences

This paper presents sufficient conditions for existence and uniqueness of a steady state equilibrium in an OLG model with non-separable preferences and analyses the implications of such assumption for the local stability of the steady state equilibrium. The conditions for a stable solution are derived under the assumption that habits are transmitted both across and within generations. Under this assumption, the paper shows that monotonic convergence to the steady state is not always assured. The paper thus proves that also the optimal solution may be affected by instability and explosive dynamics, under particular conditions on the relevant parameters.

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